Journal of Basic and Applied Engineering Research
p-ISSN: 2350-0077; e-ISSN: 2350-0255; Volume 4, Issue 6; July-September, 2017, pp. 468-473

http://www.krishisanskriti.org/Publication.html

DUAL FACTORABLE SURFACES IN THE THREE DIMENSIONAL SIMPLY ISOTROPIC SPACE I_3^1

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Abstract

A factorable surface arises as a graph of a function $w = f(u)g(v)$. In this paper, we study the dual factorable surfaces in three dimensional simply isotropic space. We classify dual factorable surface with constant dual isotropic mean curvature or constant dual isotropic Guassian curvature.

1. Introduction

Projective spaces enjoy a principle of duality, for example in projective 3-spaces, points are dual to planes and vice versa, straight lines are dual to straight lines and inclusions are reversed. However, duality cannot be applied to metric quantities of Euclidean geometry. This is dierent in isotropic geometry, which have a metric duality which may be realized by the polarity with respect to the isotropic unit sphere

$$: \quad z = \frac{1}{2}(x^2 + y^2):$$

It maps a point $p = (p_1; p_2; p_3)$ to the plane P with equation $z = p_1x + p_2y \quad p_3$. Points p, q with i-distance d are mapped to planes P, Q with i-angle d and vice versa. Parallel points correspond in the duality to parallel planes.

A surface $x: z = h(u; v)$, seen as set of contact elements (points plus tangent planes) corresponds to a surface x, parameterized by

$$\begin{cases} x = h_u(u; v); \\ y = h_v(u; v); \\ z = uh_u(u; v) + vh_v(u; v) \quad h(u; v): \end{cases}$$

Contact elements along isotropic principal curvature lines of M and M correspond in the duality. Note that M may have singularities which correspond to parabolic surface points of M $(K = 0)$. This is reected in the following relations between the isotropic curvature measures of dual surface pairs:

$$K = \frac{1}{K}; \quad H = \frac{H}{K};$$

where $H(K)$ is the isotropic mean(Gaussian) curvature and $H(K)$ is the dual isotropic mean(Gaussian) curvature. Thus, the dual isotropic minimal surface is also isotropic minimal ([8, 9]).

In this paper, we extended the notion of duality to translation surfaces in isotropic spaces and obtain

2000 Mathematics Subject Classication. Primary 53A35, 53B30, 53A40.
Key words and phrases. Dual surface, simply isotropic space, factorable surface.

the classication results for dual isotropic curvatures of translation surfaces.

2. Preliminaries

Motions and metric Isotropic geometry is based on the following group G_6 of ane transformations $(x;y;z) \to (x^0;y^0;z^0)$ in R^3;

$$\left.\begin{aligned} x^0 &= a + x\cos\phi - y\sin\phi \\ y^0 &= b + x\sin\phi + y\cos\phi \\ z^0 &= c + c_1x + c_2y + z \end{aligned}\right\};$$

where $a;b;c;c_1;c_2;\phi \in R$. Such ane transformations are called isotropic congruence transformations or isotropic motions. We see that isotropic motions appear as Euclidean motions (a translation and a rotation) in the projection onto the xyplane the result of this projection, $P = (x;y;z) \to P^0 = (x;y;0)$ is called the "top view" ([9, 8, 16]). Hence, an isotropic motion is composed of a Euclidean motion in the xyplane and an ane shear transformation in the z direction.

On the other hand, the isotropic distance of two points $P = (x_1;y_1;z_1)$ and $Q = (x_2;y_2;z_2)$ is dened as the Euclidean distance of the top views, i.e.,

$$d(P;Q)_i = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}:$$

Let $X = (x_1;y_1;z_1)$ and $Y = (x_2;y_2;z_2)$ be vectors in I^1_3. The isotropic inner product of X and Y is dened by

$$\langle X;Y\rangle_i = \begin{cases} z_1z_2; & \text{if } x_i = y_i = 0 \\ x_1x_2 + y_1y_2; & \text{if otherwise} \end{cases}$$

We call a vector of the form $X = (0;0;z)$ in I^1_3 an isotropic vector, and a non-isotropic vector otherwise. Consider a C^r-surface M, $r \geq 1$, in I^1_3 parameterized by

$$x(u;v) = (x(u;v);y(u;v);z(u;v)):$$

A surface M immersed in I^1_3 is called admissible if it has no isotropic tangent planes. We restrict our framework to admissible regular surfaces ([1, 12, 9, 16]).

For such a surface, the coecients $E;F;G$ of its rst fundamental form are calculated with respect to the induced metric and the coecients $L;M;N$ of the second fundamental form, with respect to the normal vectore ld of a surface which is always completely isotropic. The rst and the second fundamental form of M are dened by

$$\begin{aligned} I &= Edu^2 + Fdudv + Gdv^2; \\ II &= Ldu^2 + Mdudv + Ndv^2; \end{aligned}$$

where

$$E = \langle x_u;x_u\rangle_i;\; F = \langle x_u;x_v\rangle_i;\; G = \langle x_v;x_v\rangle_i;$$

$$L = \frac{\det(x_u;x_v;x_{uu})}{\sqrt{EG - F^2}};$$

$$M = \frac{\det(x_u;x_v;x_{uv})}{\sqrt{EG - F^2}};$$

$$N = \frac{\det(x_u;x_v;x_{vv})}{\sqrt{EG - F^2}}:$$

Since $EG - F^2 > 0$, for the function in the denominator we often put $W^2 = EG - F^2$. The isotropic unit normal vectore ld is given by $U = (0;0;1)$: The isotropic Gaussian curvature K and the isotropic mean curvature H are dened by

$$K = \frac{LN - M^2}{EG - F^2}; \quad 2H = \frac{EN - 2FM + GL}{EG - F^2}:$$

The surface M is said to be isotropic at (resp. isotropic minimal) if K (resp. H) vanishes ([1, 4, 9, 11, 12, 13]).

We conne our discussion to regular surfaces x without isotropic tangent planes. Thus, we may write in explicit form,

$$x: \; w = h(u; v):$$

3. Factorable Surfaces in I_3^1

A surface in Euclidean 3-space is called as a factorable or homothetical if the graph ofi ts surface is associated with $w = f(u)g(v)$. Their classication in E^3 with constant Gaussain and mean curvatures are obtained in [7, 17]. Zong et al. [18] generalized factorable surfaces to ane factorable surfaces in E^3 as the graph of the function

$$w = f(u)g(v + cu); \; 6 = 0:$$

In Minkowski space E^3, factorable surfaces arises as a graph of functions

$$\varphi_1 : w = f(u)g(v); \; \varphi_2 : v = f(u)g(w);$$
$$\varphi_3 : u = f(v)g(w)$$

There are six dierent classes of factorable surfaces in E_1^3 with respect to the character of directions [10]. Their classication with constant Gaussian and mean curvature are obtained in [10, 15].

In addition to E^3 and E_1^3, there are two types of factorable surfaces in 3-dimensional simply isotropic space I_3^1 arising from the product of uv plane and the isotropic w direction with degenerate metric [9]. Due to the isotropic axes in I_3^1 the factorable surface φ_1 is dierent from φ_2 and φ_2. We call φ_1 as factorable surface of type 1 [2] and is parametrized as

$$x(u; v) = (u; v; f(u)g(v)): \tag{1}$$

The dual isotropic Gaussian and dual isotropic mean curvature of φ_1 can be easily found as

$$\begin{aligned} K &= \frac{1}{f'^2 g'^2 + fgf''^2 g''^2}; \\ H &= \frac{fg'' + gf''}{2\, f'^2 g'^2 + fgf''^2 g''^2} \end{aligned} \tag{2}$$

The graphs of φ_2 and φ_3 are locally isometric and, upto to a sign, have same second fundamental form. Therefore upto a sign, they have same Gaussian and mean curvature, so φ_2 or φ_3 is called as factorable surfaces of type 2 [2] and is parametrized as

$$\begin{aligned} x(u; w) &= (u; f(u)g(w); w); \\ x(v; w) &= (f(v)g(w); v; w): \end{aligned} \tag{3}$$

The dual isotropic Gaussian curvature and dual isotropic mean curvature of φ_3 can be easily found as

$$H = \frac{f^2 g' \quad f'g^2 + 1 \quad g'' + gg'^2 \quad ff'' \quad 2f'^2}{2\, f'^2 g'^2 + fgf''^2 g''^2} \tag{4}$$

$$K = \frac{(fg')^4}{f'^2 g'^2 + fgf''^2 g''^2}: \tag{5}$$

In this paper, we nd the explicit forms of dual isotropic factorable surfaces of type-1 with constant Gaussian and mean curvatures. A similar discussion of type-2 can be a problem of future research. Noting that throughout this paper c_i^0s will denote non-zero reals and d_i^0s as any real number.

4. Dual isotropic factorable surfaces of type-1 in I_3^1

Suppose that the dual isotropic Gaussian K curvature of φ_1 is a non-zero constant, then from (2), we have

$$f'^2 g'^2 + fgf''^2 g''^2 = \frac{1}{K} \tag{6}$$

Case 1: Suppose $f = c_1 u + d_1$ be linear, then from (6), we have

$$g = \frac{v}{c_1 \sqrt{K}} + d_2:$$

Case 2: Suppose f be non-linear, dividing (6) by ff'', we get

$$gg'' = \frac{1}{K ff''} + \frac{f'^2 g'^2}{ff''} \tag{7}$$

Dierentiating (7) partially w.r.t. v, we get

$$gg''{}' = \frac{f'^2}{ff''}\left(g'^2\right)':$$

Since f and g are functions of two independent variables u and v, above equation can be written as

$$\frac{(gg'')'}{\left(g'^2\right)'} = \frac{f'^2}{ff''} = p; \tag{8}$$

where p is a constant. From (8), we get the following system of equations

$$\begin{aligned} gg'' - pg'^2 &= a \\ f'^2 - pff'' &= 0, \end{aligned} \tag{9}$$

for some constant a. Thus from (9), we get

$$\begin{cases} f = c_2 e^{c_1 u}; \\ g = \frac{1}{4} e^{-c_1 v - d_2}\left(d_3 + e^{2c_1(v+d_4)}\right); \\ \text{or} \\ g = \frac{1}{4} e^{-c_1 v - d_2}\left(1 + 4ae^{2c_1 v + d_5}\right) \end{cases} \tag{10}$$

for $p = 1$

or,

$$\begin{cases} f = c_2\sqrt{2u - d_1} \\ g = \sqrt{\frac{e^{2c_1}}{a} + av^2 + 2avd_2 + ad_2^2} \end{cases}; \tag{11}$$

for $p = 1$.

Hence, we have the following result.

Theorem 4.1. Let M be a factorable surface of type-1 in simply isotropic space I_3^1 with dual isotropic Gaussian curvature $K \neq 0$, then $f = c_1 u + d_1$ and $g = \frac{v}{c_1\sqrt{K}} + d_2$, or are given by (10) and (11).

Also, from (2), we see that

Corollary 4.2. There are no factorable surfaces of type-1 in I_3^1, with vanishing dual Gaussian curvature.

Now, suppose the dual isotropic mean curvature of $_1$ is a non-zero constant $H \neq 0$. Then, from (2), we have

$$\frac{f''}{f} + \frac{g''}{g} = 2H\left(\frac{f'^2 g'^2}{fg} + f''g''\right) \tag{12}$$

Case 1: Suppose $f = c_1 u + d_2$ be linear, then from (12), we get

$$(c_1 u + d_2)g'' = 2H\left(c_1^2 g'^2\right):$$

Dierentiating above equation w.r.t. u, we get

$$c_1 g'' = 0$$

This implies that g is also linear i.e., $g = c_3 v + d_4$:

Case 2: Suppose f and g are non-linear. Dierentiating (12) w.r.t. u and v, we get

$$\left(\frac{f'^2}{f}\right)'\left(\frac{g'^2}{g}\right)' = f'''g''':$$

Since f and g are functions of two independent variables u and v, the above equation can be written as

$$\frac{1}{f'''}\left(\frac{f'^2}{f}\right)' = \frac{g'''}{\left(\frac{g'^2}{g}\right)'} = p; \tag{13}$$

where p is a constant.

From (13), we get the following equations

$$f'^2 - pff'' = af; \tag{14}$$

and

$$gg'' = pg'^2 + bg; \tag{15}$$

where $a; b$ are constants. Let $a^2 + b^2 \neq 0$:

The cases for which the solutions of (14) and (15) exists are as following.

Subcase 2.1: Suppose ($p = 1; a = 1$) and ($p = 1, b = 1$), from (14) and (15), we get

$$f = c_3 \sinh^2\left(\frac{1}{2}(c_1 u + d_2)\right) \tag{16}$$

and

(17)
$$\begin{cases} g = \dfrac{e^{c_1 vd_2}\left(e^{2c_1(v+d_3)}+16c_4 e^{c_1 v+d_2}+64c_4^2\right)}{16c_4^2} \\ g = \dfrac{e^{c_1 vd_2}\left(1+16c_4 e^{c_1 v+d_2}+64c_4^2 e^{2c_1(v+d_3)}\right)}{16c_4^2} \end{cases}$$

Subcase 2.2: Suppose ($p = 1; a \neq 1$) and ($p = 1, b \neq 1$), from (14) and (15), we get

(18)
$$\begin{cases} f = \dfrac{e^{c_1 ud_2}\left(e^{c_1 u+d_2} 8ad_3\right)^2}{16c_4^2} \\ f = \dfrac{e^{c_1 ud_2}\left(1+8ad_3 e^{c_1 u+d_2}\right)^2}{16c_4^2} \end{cases}$$

and

(19)
$$\begin{cases} g = \dfrac{e^{c_1 vd_2}\left(e^{c_1 v+d_2} 8bc_3\right)^2}{16c_3^2} \\ g = \dfrac{e^{c_1 vd_2}\left(1+8bc_3 e^{c_1 u+d_2}\right)^2}{16c_3^2} \end{cases}$$

Theorem 4.3. Let M be a factorable surface of type-1 in simply isotropic space I_3^1 with dual isotropic mean curvature $H \neq 0$, then f and g are either both linear or given by (16),(17),(18) and (19).

Now, suppose that the dual isotropic mean curvature of factorable surface of type-2 vanishes identically, then from (2), we have

$$fg'' + gf'' = 0$$

The above equation can be written as

$$\frac{f''}{f} = \frac{g''}{g} = p; \tag{20}$$

where p is some constant. From (20), we get

$$f = c_1 e^{\sqrt{p}u} + c_2 e^{\sqrt{p}u}; \tag{21}$$

and

$$g = c_3 \cos(\sqrt{p}v) + c_4 \sin(\sqrt{p}v). \tag{22}$$

Therefore, we have the following result.

Theorem 4.4. Let M be a factorable surface of type-1 in simply isotropic space I_3^1 with dual isotropic mean curvature $H = 0$, then f and g are given by (21) and (22) .

Acknowledgment

We are very thankful to the fourth author, Manzoor Ahmad Lone for his help in handling softwares(mathematica, matlab) which we used to solve various dierential equations.

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